EJERCICIO 1

Probar que det $e^A = e^{Tr\,A}$

Hipótesis: A es diagonalizable, de modo que $A = M\,S\,M^{-1}$

S es la matriz diagonal: $\begin{pmatrix} \lambda_1 & & \cdots & & 0 \\ & \lambda_2 & & & \\ \vdots & & \ddots & \vdots & \\ & & \cdots & \lambda_{N-1} & \\ 0 & & & & \lambda_N \end{pmatrix}$ donde λ_i son los autovalores.

M la construimos con los autovectores normalizados, de modo que det (M) = 1

$$e^A = e^{M\,S\,M^{-1}} = I + (M\,S\,M^{-1}) + \frac{1}{2!}(M\,S\,M^{-1})^2 + \cdots$$
$$= (M\,M^{-1}) + (M\,S\,M^{-1}) + \frac{1}{2!}(M\,S\,M^{-1})^2 + \cdots$$

$(M\,S\,M^{-1})^2 = (M\,S\,M^{-1})\,(M\,S\,M^{-1}) = M\,S^2\,M^{-1}$

$$e^A = M\left\{I + S + \frac{1}{2!}S^2 + \cdots\right\}M^{-1}$$

$$e^A = M\begin{pmatrix} \left(1+\lambda_1+\frac{1}{2!}{\lambda_1}^2+\cdots\right) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \left(1+\lambda_N+\frac{1}{2!}{\lambda_N}^2+\cdots\right) \end{pmatrix}M^{-1}$$

$$e^A = M\begin{pmatrix} e^{\lambda_1} & & \cdots & & 0 \\ & e^{\lambda_2} & & & \\ \vdots & & \ddots & \vdots & \\ & & \cdots & e^{\lambda_{N-1}} & \\ 0 & & & & e^{\lambda_N} \end{pmatrix}M^{-1}$$

det (AB) = det A x det B

$$\det e^A = \det M \times \det\begin{pmatrix} e^{\lambda_1} & & \cdots & & 0 \\ & e^{\lambda_2} & & & \\ \vdots & & \ddots & \vdots & \\ & & \cdots & e^{\lambda_{N-1}} & \\ 0 & & & & e^{\lambda_N} \end{pmatrix} \times \det M^{-1}$$

$$\det e^A = 1 \times \left(e^{\lambda_1} e^{\lambda_2} \ldots e^{\lambda_N}\right) \times 1 = e^{\lambda_1+\lambda_2+\cdots+\lambda_N}$$

En una matriz A diagonalizable $Tr\,A = \sum_1^N \lambda_i$

Entonces:

$$\mathbf{det}\,\boldsymbol{e^A} = \boldsymbol{e^{Tr\,A}}$$

EJERCICIO 2

Probar que si el operador § cumple con las propiedades

i) de bilinealidad;

y iii) tal que A § A = 0; ∀ A

Entonces se cumple la propiedad de antisimetría: A § B = - B § A

[1] (A + B) § A = A § A + B § A = B § A

[2] (A + B) § B = A § B + B § B = A § B

Haciendo [1] + [2]

(A + B) § A + (A + B) § B = B § A + A § B

Pero, por la propiedad i): C § A + C § B = C § (A + B)

Entonces

(A + B) § (A + B) = B § A + A § B

Y por la propiedad iii):

0 = B § A + A § B

A § B = - B § A

EJERCICIO 3

Dado el espacio vectorial $\boldsymbol{V = \{ax^2 + bx + c;\ +\ ;\ \cdot\}}$

Verificar que el operador § definido como

$$(a_1 x^2 + a_2 x + a_3) \ \S\ (b_1 x^2 + b_2 x + b_3) = (a_2 b_3 - a_3 b_2) x^2 + (a_3 b_1 - a_1 b_3) x + (a_1 b_2 - a_2 b_1)$$

Cumple las propiedades

- **i) Bilinealidad**
- **ii) Identidad de Jacobi**
- **iii) A § A = 0; ∀ A**

i) Bilinealidad

$(\alpha A + \beta B) \ \S\ C = \alpha A \ \S\ C + \beta B \ \S\ C$

$$[\alpha (a_1 x^2 + a_2 x + a_3) + \beta (b_1 x^2 + b_2 x + b_3)] \ \S\ (c_1 x^2 + c_2 x + c_3) =$$

$$= [(\alpha a_1 + \beta b_1) x^2 + (\alpha a_2 + \beta b_2) x + (\alpha a_3 + \beta b_3)] \ \S\ (c_1 x^2 + c_2 x + c_3) =$$

$$= [(\alpha a_2 + \beta b_2) c_3 - (\alpha a_3 + \beta b_3) c_2] x^2 + [(\alpha a_3 + \beta b_3) c_1 - (\alpha a_1 + \beta b_1) c_3] x +$$
$$+ [(\alpha a_1 + \beta b_1) c_2 - (\alpha a_2 + \beta b_2) c_1] =$$

$= (\alpha a_2 c_3 - \alpha a_3 c_2) x^2 + (\alpha a_3 c_1 - \alpha a_1 c_3) x + (\alpha a_1 c_2 - \alpha a_2 c_1) +$
$+ (\beta b_2 c_3 - \beta b_3 c_2) x^2 + (\beta b_3 c_1 - \beta b_1 c_3) x + (\beta b_1 c_2 - \beta b_2 c_1) =$

$= \alpha (a_2 c_3 - a_3 c_2) x^2 + \alpha (a_3 c_1 - a_1 c_3) x + \alpha (a_1 c_2 - a_2 c_1) +$
$+ \beta (b_2 c_3 - b_3 c_2) x^2 + \beta (b_3 c_1 - b_1 c_3) x + \beta (b_1 c_2 - b_2 c_1) =$

$= \alpha [(a_2 c_3 - a_3 c_2) x^2 + (a_3 c_1 - a_1 c_3) x + (a_1 c_2 - a_2 c_1)] +$
$+ \beta [(b_2 c_3 - b_3 c_2) x^2 + (b_3 c_1 - b_1 c_3) x + (b_1 c_2 - b_2 c_1)] =$

$= \alpha A § C + \beta B § C$

C § (α A + β B) = α C § A + β C § B

$(c_1 x^2 + c_2 x + c_3) § [\alpha (a_1 x^2 + a_2 x + a_3) + \beta (b_1 x^2 + b_2 x + b_3)] =$

$= (c_1 x^2 + c_2 x + c_3) § [(\alpha a_1 + \beta b_1) x^2 + (\alpha a_2 + \beta b_2) x + (\alpha a_3 + \beta b_3)] =$

$= [c_2 (\alpha a_3 + \beta b_3) - c_3 (\alpha a_2 + \beta b_2)] x^2 + [(c_3 (\alpha a_1 + \beta b_1) - c_1 (\alpha a_3 + \beta b_3)] x +$
$+ [c_1 (\alpha a_2 + \beta b_2) - c_2 (\alpha a_1 + \beta b_1)] =$

$= (\alpha c_2 a_3 - \alpha c_3 a_2) x^2 + (\alpha c_3 a_1 - \alpha c_1 a_3) x + (\alpha c_1 a_2 - \alpha c_2 a_1) +$
$+ (\beta c_2 b_3 - \beta c_3 b_2) x^2 + (\beta c_3 b_1 - \beta c_1 b_3) x + (\beta c_1 b_2 - \beta c_2 b_1) =$

$= \alpha (c_2 a_3 - c_3 a_2) x^2 + \alpha (c_3 a_1 - c_1 a_3) x + \alpha (c_1 a_2 - c_2 a_1) +$
$+ \beta (c_2 b_3 - c_3 b_2) x^2 + \beta (c_3 b_1 - c_1 b_3) x + \beta (c_1 b_2 - c_2 b_1) =$

$= \alpha [(c_2 a_3 - c_3 a_2) x^2 + (c_3 a_1 - c_1 a_3) x + (c_1 a_2 - c_2 a_1)] +$
$+ \beta [(c_2 b_3 - c_3 b_2) x^2 + (c_3 b_1 - c_1 b_3) x + (c_1 b_2 - c_2 b_1)] =$

$= \alpha C § A + \beta C § B$

ii) Identidad de Jacobi

(A § B) § C + (C § A) § B + (B § C) § A = 0

$[(a_1 x^2 + a_2 x + a_3) § (b_1 x^2 + b_2 x + b_3)] § (c_1 x^2 + c_2 x + c_3) + [(c_1 x^2 + c_2 x + c_3) § (a_1 x^2 + a_2 x + a_3)] § (b_1 x^2 + b_2 x + b_3) + [(b_1 x^2 + b_2 x + b_3) § (c_1 x^2 + c_2 x + c_3)] § (a_1 x^2 + a_2 x + a_3) = 0$

$= [(a_2 b_3 - a_3 b_2) x^2 + (a_3 b_1 - a_1 b_3) x + (a_1 b_2 - a_2 b_1)] § (c_1 x^2 + c_2 x + c_3) +$

$+ [(c_2 a_3 - c_3 a_2) x^2 + (c_3 a_1 - c_1 a_3) x + (c_1 a_2 - c_2 a_1)] § (b_1 x^2 + b_2 x + b_3) +$

$+ [(b_2 c_3 - b_3 c_2) x^2 + (b_3 c_1 - b_1 c_3) x + (b_1 c_2 - b_2 c_1)] § (a_1 x^2 + a_2 x + a_3) =$

$= [(a_3 b_1 - a_1 b_3) c_3 - (a_1 b_2 - a_2 b_1) c_2] x^2 + [(a_1 b_2 - a_2 b_1) c_1 - (a_2 b_3 - a_3 b_2) c_3] x +$
$+ [(a_2 b_3 - a_3 b_2) c_2 - (a_3 b_1 - a_1 b_3) c_1] +$

$+ [(c_3 a_1 - c_1 a_3) b_3 - (c_1 a_2 - c_2 a_1) b_2] x^2 + [(c_1 a_2 - c_2 a_1) b_1 - (c_2 a_3 - c_3 a_2) b_3] x +$
$+ [(c_2 a_3 - c_3 a_2) b_2 - (c_3 a_1 - c_1 a_3) b_1] +$

$+ [(b_3 c_1 - b_1 c_3) a_3 - (b_1 c_2 - b_2 c_1) a_2] x^2 + [(b_1 c_2 - b_2 c_1) a_1 - (b_2 c_3 - b_3 c_2) a_3] x +$
$+ [(b_2 c_3 - b_3 c_2) a_2 - (b_3 c_1 - b_1 c_3) a_1] =$

$= [\ (a_3\, b_1 - a_1\, b_3)\, c_3 - (a_1\, b_2 - a_2\, b_1)\, c_2 + (c_3\, a_1 - c_1\, a_3)\, b_3 - (c_1\, a_2 - c_2\, a_1)\, b_2 + (b_3\, c_1 - b_1\, c_3)\, a_3$
$- (b_1\, c_2 - b_2\, c_1)\, a_2\ ]\, x^2 +$
$+ [\ (a_1\, b_2 - a_2\, b_1)\, c_1 - (a_2\, b_3 - a_3\, b_2)\, c_3 + (c_1\, a_2 - c_2\, a_1)\, b_1 - (c_2\, a_3 - c_3\, a_2)\, b_3 + (b_1\, c_2 - b_2\, c_1)\, a_1$
$- (b_2\, c_3 - b_3\, c_2)\, a_3\ ]\, x +$
$+ [\ (a_2\, b_3 - a_3\, b_2)\, c_2 - (a_3\, b_1 - a_1\, b_3)\, c_1 + (c_2\, a_3 - c_3\, a_2)\, b_2 - (c_3\, a_1 - c_1\, a_3)\, b_1 + (b_2\, c_3 - b_3\, c_2)\, a_2$
$- (b_3\, c_1 - b_1\, c_3)\, a_1\ ] =$

$= [\ a_3\, b_1\, c_3 - a_1\, b_3\, c_3 - a_1\, b_2\, c_2 + a_2\, b_1\, c_2 + c_3\, a_1\, b_3 - c_1\, a_3\, b_3 - c_1\, a_2\, b_2 + c_2\, a_1\, b_2 + b_3\, c_1\, a_3 - b_1\, c_3\, a_3$
$- b_1\, c_2\, a_2 + b_2\, c_1\, a_2\ ]\, x^2 +$
$+ [\ a_1\, b_2\, c_1 - a_2\, b_1\, c_1 - a_2\, b_3\, c_3 + a_3\, b_2\, c_3 + c_1\, a_2\, b_1 - c_2\, a_1\, b_1 - c_2\, a_3\, b_3 + c_3\, a_2\, b_3 + b_1\, c_2\, a_1 - b_2\, c_1\, a_1$
$- b_2\, c_3\, a_3 + b_3\, c_2\, a_3\ ]\, x +$
$+ [\ a_2\, b_3\, c_2 - a_3\, b_2\, c_2 - a_3\, b_1\, c_1 + a_1\, b_3\, c_1 + c_2\, a_3\, b_2 - c_3\, a_2\, b_2 - c_3\, a_1\, b_1 + c_1\, a_3\, b_1 + b_2\, c_3\, a_2 - b_3\, c_2\, a_2$
$- b_3\, c_1\, a_1 + b_1\, c_3\, a_1\ ] =$

$= 0$

(A § B) § C + (C § A) § B + (B § C) § A = 0

iii) A § A = 0; ∀ A

Haciendo $b_1 = a_1$; $b_2 = a_2$ y $b_3 = a_3$

$(a_1\, x^2 + a_2\, x + a_3)\ §\ (a_1\, x^2 + a_2\, x + a_3) = (a_2\, a_3 - a_3\, a_2)\, x^2 + (a_3\, a_1 - a_1\, a_3)\, x + (a_1\, a_2 - a_2\, a_1) = 0$

A § A = 0